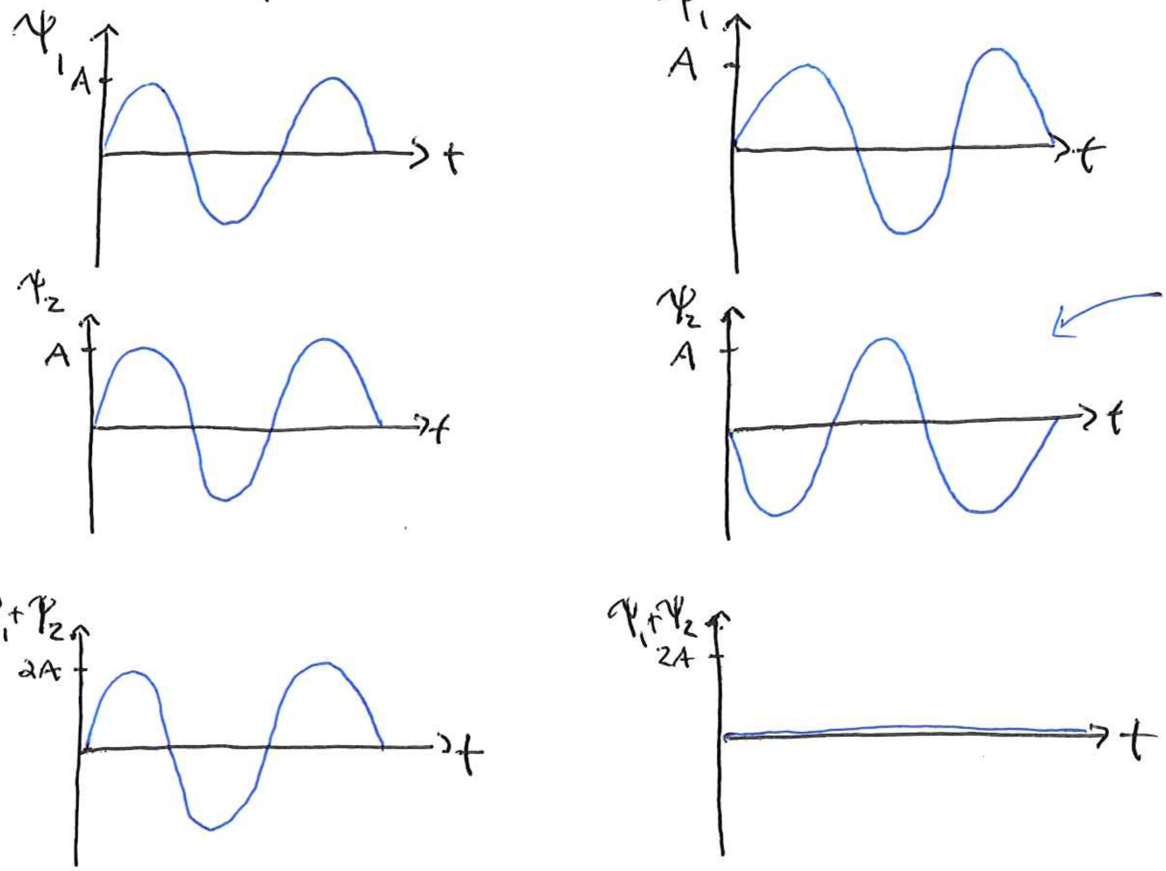
Classical & Quantum Waves

Interference of waves

- Consider two monochromatic (= single frequency/mode) sinusoidal waves ψ_1, ψ_2 that originate from same source, traverse ~~different~~ path lengths difference s, then overlap again: [s = (path 1) − (path 2)]

- Two important cases:

"in phase" "out-of-phase"



π phase shift, which reverses amplitude

Constructive interference	Destructive interference
$s = n\lambda$, $n = 0, \pm 1, \pm 2, \ldots$	$s = (n+\frac{1}{2})\lambda$, $n = 0, \pm 1, \pm 2, \ldots$
$\phi = 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$	$\phi = (2n+1)\pi$, $n = 0, \pm 1, \pm 2, \ldots$

path length difference

ϕ: phase difference associated w/ path length difference

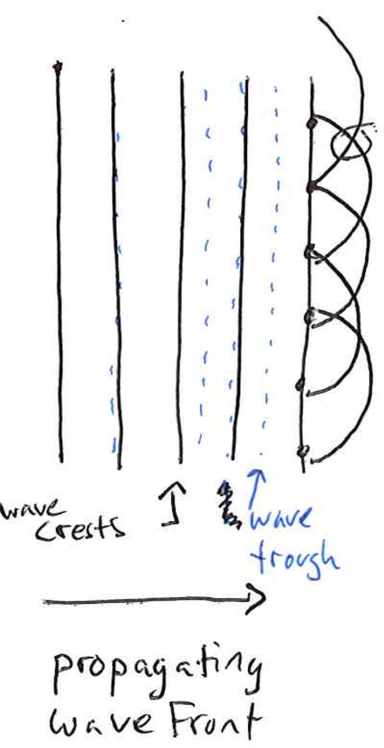
- For other values of path length difference δ, the resulting interference will be in between the two extremes of perfect constructive and perfect destructive interference

- Why should we care? Many examples & applications
  - Noise cancelling headphones
  - precision length measurements (laser interferometer)

    [If map out constructive ↔ destructive for known λ, can measure δ]

// ~~Hyugens~~ Huygens' principle

- States that every point on a wavefront acts as a source of spherical secondary wavelets that mutually interfere



← where wavelets constructively interfere (ie, where crests align) ends up being the next crest in the overall propagating wave front

- radius of each arc is $v\Delta t$ [v: wave velocity]

→ can consider a plane wave as ∞ superposition of spherical waves!

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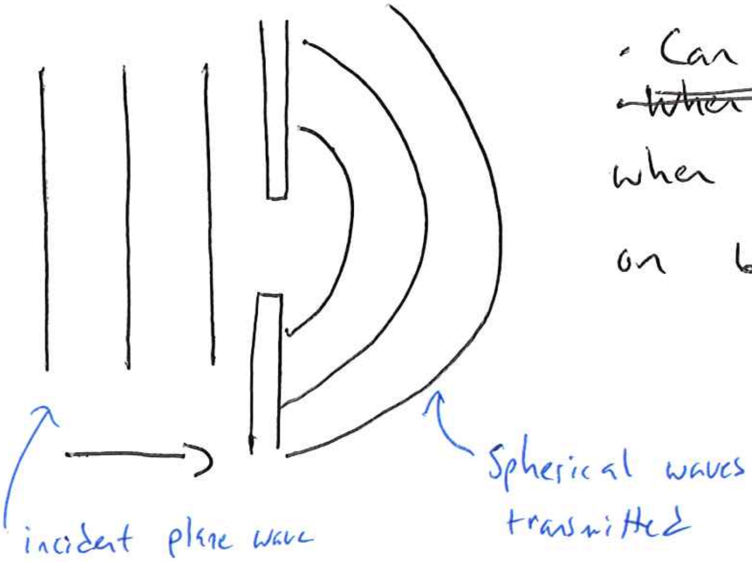


- Can see this clearly ~~when to~~ when consider plane wave incident on barrier w/ narrow operature

→ Explains why sound can "bend" around corners

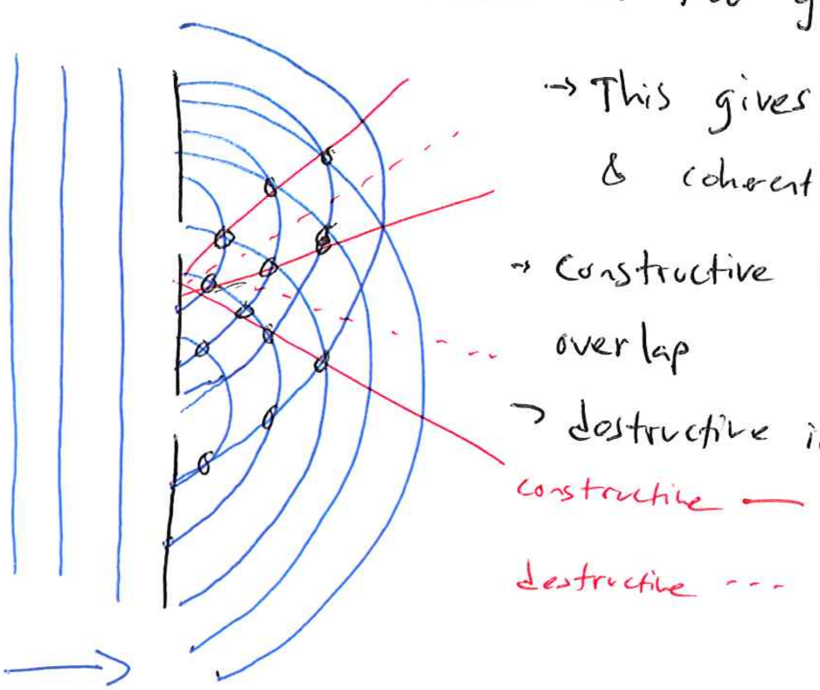
- What about spherical waves traveling in reverse direction from wavefront?

→ Huygens' principle conveniently ignores this, and it also doesn't happen in reality. Kirchoff came up w/ an exact theory for all this (but we don't need for this class)

Young's double slit experiment

[Young also deciphered Rosetta Stone!]

- Consider now a barrier w/ two gaps



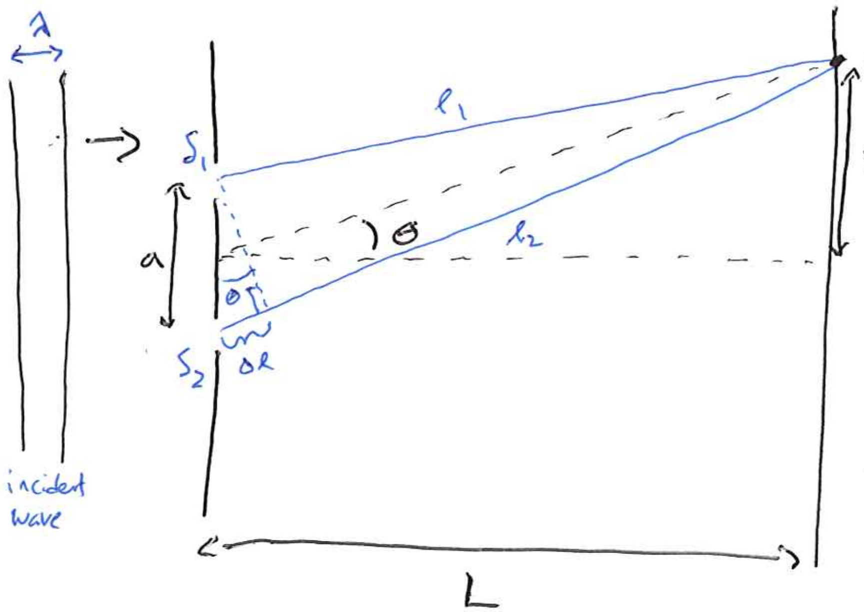
→ This gives two sources of monochromatic & coherent waves!

→ Constructive interference where two crests overlap

→ destructive interference where crest meets trough

- This means that if we were to put a screen very far away, we would detect regions of constructive & destructive interference (say, if working w/ light)

- Let's formalize this:



- Assume $L >> a$ (typically, $L \sim 10$ m, $a \approx 0.5$ mm)
- Common incident wave ensures well-defined phase relation b/t two waves emenating from slits
- Assume $L >> x$ (small ∡)

- We want to know the superposition of the two wavefronts at point P

$$R = A[\cos(\omega t - k\ell_1) + \cos(\omega t - k\ell_2)]$$ [R = resulting amplitude from interfering waves]

$$\downarrow$$

$$= 2A\cos[\omega t - k(\ell_2+\ell_1)/2]\cos[k(\ell_2-\ell_1)/2]$$ [see attached proof]

- We are interested in the intensity, $I \propto R^2$

$$I \propto 4A^2 \underbrace{\cos^2[\omega t - k(\ell_1+\ell_2)/2]}_{=1/2} \cos^2[k(\ell_2-\ell_1)/2]$$

- For light, $\omega \sim 10^{15}$ Hz → undetectably fast, $\cos^2[\;] = \frac{1}{2}$

avg. intensity

$$\Rightarrow \bar{I} = I_0 \cos^2[k(\ell_2-\ell_1)/2]$$

$I_0 = 2A^2$ [the factor of $\frac{1}{2}$ turns 4 → 2]

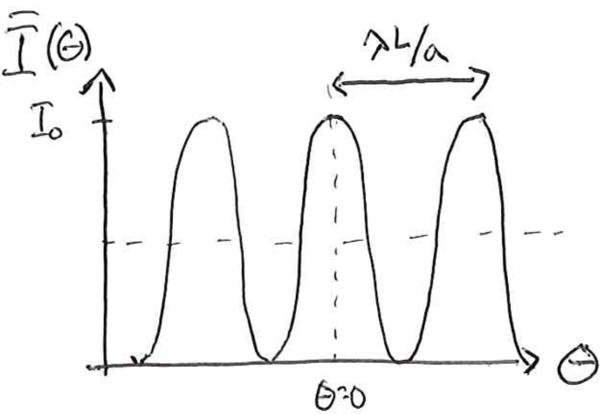
path length difference $\ell_2 - \ell_1$ matters!

$\Delta \ell = \ell_2 - \ell_1 \approx a \sin\theta$ [see graph previous page]

(small angle approx)

$\xrightarrow{\text{small } \measuredangle} \simeq a\theta$

$$\Rightarrow \boxed{\bar{I}(\theta) = I_0 \cos^2\left(\frac{ka\theta}{2}\right) = I_0 \cos^2\left(\frac{\pi a \theta}{\lambda}\right)} \quad k = \frac{2\pi}{\lambda}$$



Interference Fringes

→ If not coherent, would see $I_0/2$ everywhere

- Intensity maxima occur when $\theta = \frac{n\lambda}{a}$, $n = 0, \pm 1, \ldots$

$$\xrightarrow{\text{small } \measuredangle} x = L\theta = n\frac{\lambda L}{a}, \quad n = 0, \pm 1, \ldots$$

- Minima: $x = \left(n + \frac{1}{2}\right)\frac{\lambda L}{a}$, $n = 0, \pm 1, \ldots$

- Distance b/t adjacent fringes is $x_{n+1} - x_n = \frac{\lambda L}{a}$ for all n

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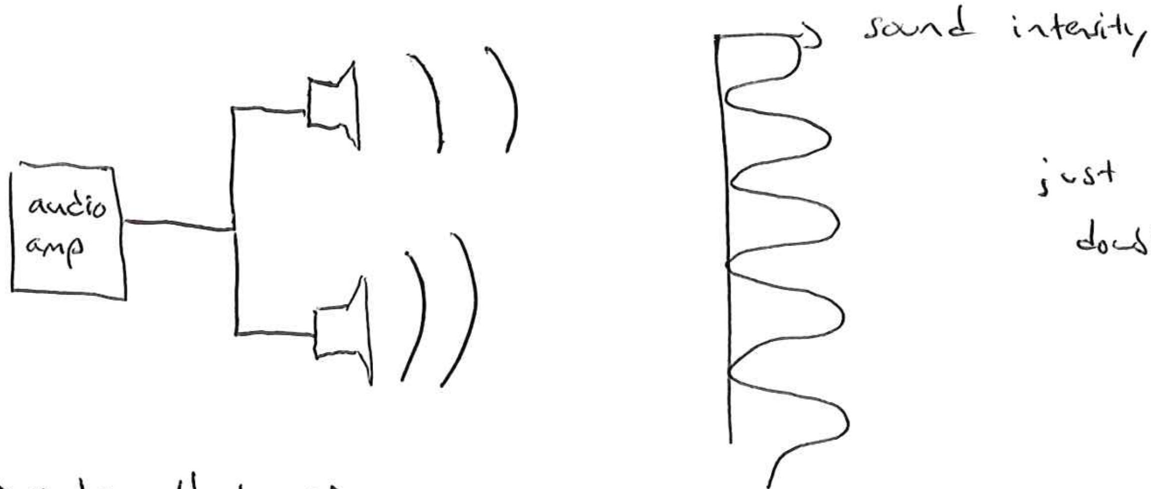
Examples of interference

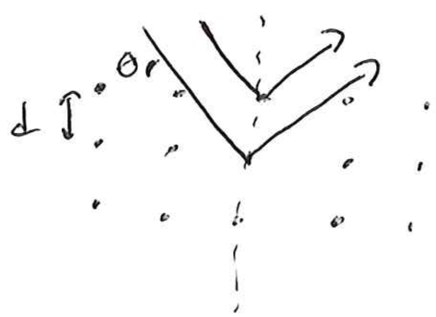
- Two speakers driven by <u>same</u> amplifier



just like double slit

→ note that if each speaker driven by <u>different</u> amp, no interference b/c ~~will have different phase~~ phase relationship b/t two speakers will not be constant

- X-ray diffraction & crystallography



<u>Bragg condition</u> for constructive interference

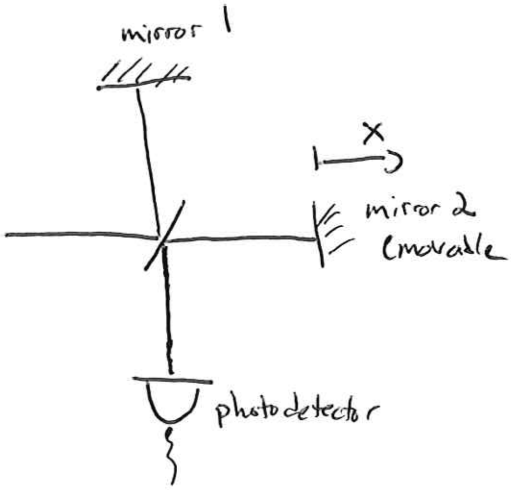
$$\boxed{2d\sin\theta = n\lambda} \qquad n = \pm 1, \pm 2, \ldots$$

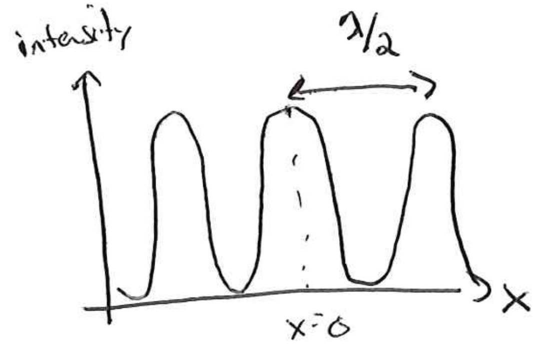
→ deduce crystal structure from angles of maxima

→ used to discover double helix of DNA!

→

· Michelson interferometer





- Can measure wavelength of light, or if λ known make sensitive length measurements
- ~~use~~
- Used to disprove existence of ether (1887)
- Used to detect gravitational waves at LIGO

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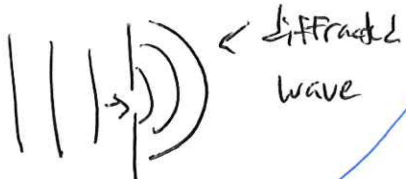
Diffraction

- Describes "bending" of waves around the corners of an obstacle or through an aperature → described by Huygens' principle discussed earlier

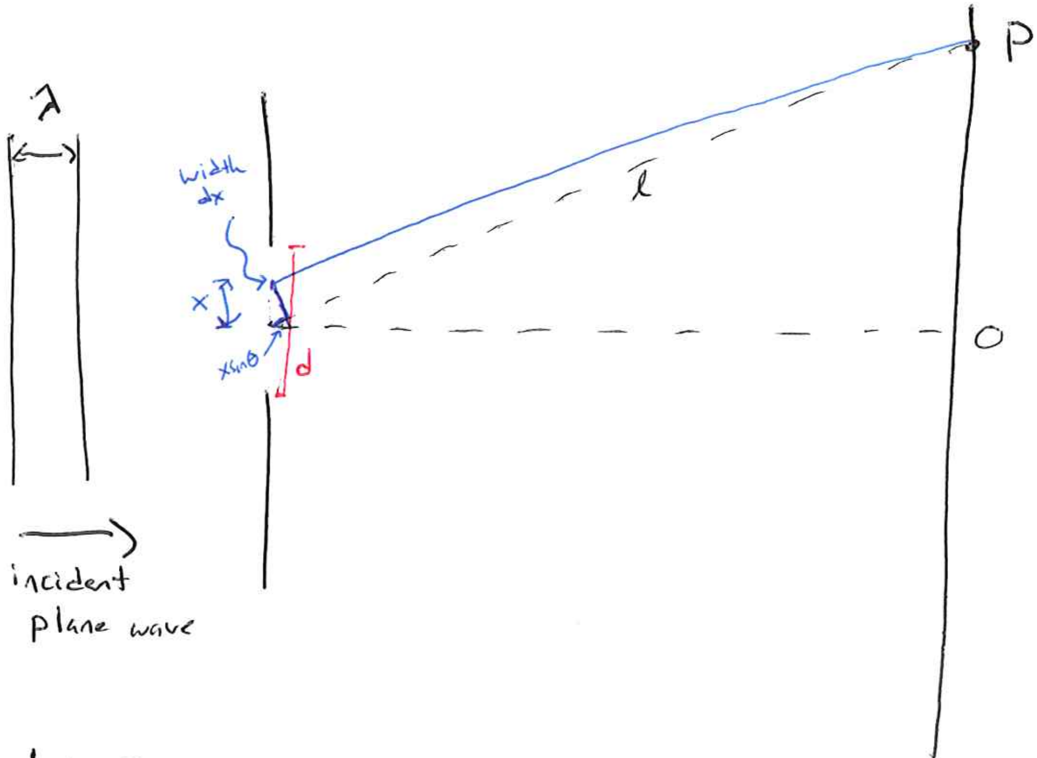


why we can hear sound around a corner

- Degree of spreading out of wave, depends on ratio of λ to size d of aperature ~ λ/d (shown below)

Diffraction from single slit

- When discussing Young's double slit experiment earlier, we assumed width of slits very narrow, so we could assume path across width of each slit to a point P on screen was equal
- In reality, slits have finite width and there will thus be a length difference and hence phase difference when wavelets hit screen



· Interference at P results from superposition of wavelets that emerge from infinitesimal strips dx with amplitude:

$$dR = \alpha \, dx \cos\Big[\omega t - k\underbrace{(\ell - x\sin\theta)}_{\text{total length to point P from a given point x}}\Big]$$

α: constant

ω: angular freq.
k: wavenumber

~~length difference b/t wavelet generated at dx at position x vs origin~~

· The amplitude at P resulting from all wavelets from all strips dx is:

$$R = \int_{-d/2}^{d/2} \alpha \, dx \cos\left[\omega t - k(\ell - x\sin\theta)\right]$$

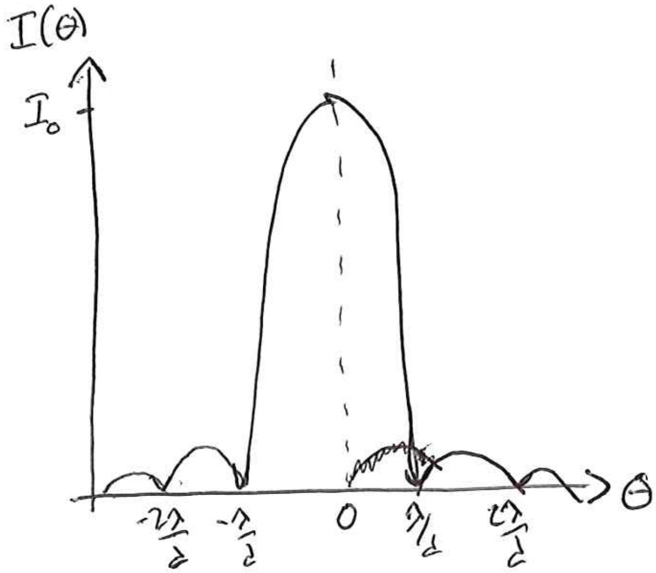
[no proof]

$$= \frac{\alpha d}{\left(\frac{kd}{2}\right)\sin\theta} \sin\left[\left(\frac{kd}{2}\sin\theta\right)\right]\cos(\omega t - k\ell)$$

$$\text{Intensity} = R^2 \Rightarrow I = \alpha^2 d^2 \underbrace{\cos^2(\omega t - k\ell)}_{=\frac{1}{2} \text{ on avg } \omega \text{ fast}} \frac{\sin^2\left[\left(\frac{kd}{2}\sin\theta\right)\right]}{\left[\left(\frac{kd}{2}\right)\sin\theta\right]^2}$$

$$\rightarrow \bar{I}(\theta) = \underbrace{I_0}_{=\frac{\alpha^2 d^3}{2}} \frac{\sin^2\left[\left(\frac{kd}{2}\right)\sin\theta\right]}{\left[\left(\frac{kd}{2}\right)\sin\theta\right]^2}$$

$\rightarrow$ (sinc function)^2 $\sim \left(\frac{\sin x}{x}\right)^2$

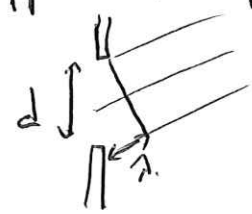


First minimum at: $\frac{kd}{2}\sin\theta = \pm\pi$

$\rightarrow \sin\theta = \pm\frac{\lambda}{d}$

Dependence of diffraction on $\frac{\lambda}{d}$!

$\rightarrow$ 1st minimum appear for path difference λ across slit



For $\lambda \ll d \xrightarrow{\sin\theta\sim\theta} \boxed{\theta = \pm\frac{\lambda}{d}}$

Trigonometric Identity Proof

Show that: R = A[cos(ωt − kl1) + cos(ωt − kl2)] = 2A cos[ωt − k(l2 + l1)/2)] cos[k(l2 − l1)/2]

To prove that:

$$R = A[\cos(\omega t - kl_1) + \cos(\omega t - kl_2)] = 2A\cos\left(\omega t - k\frac{l_2 + l_1}{2}\right)\cos\left(k\frac{l_2 - l_1}{2}\right),$$

we'll use the trigonometric identity for the sum of cosines:

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).$$

Step-by-Step Solution

1. **Define the expressions for x and y:**

$$x = \omega t - kl_1 \quad \text{and} \quad y = \omega t - kl_2.$$

2. **Apply the sum of cosines identity:** Using $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$, we substitute x and y:

$$\cos(\omega t - kl_1) + \cos(\omega t - kl_2) = 2\cos\left(\frac{(\omega t - kl_1) + (\omega t - kl_2)}{2}\right)\cos\left(\frac{(\omega t - kl_1) - (\omega t - k}{2}\right.$$

3. **Simplify the terms inside the cosines:**
   - For the first cosine term:

$$\frac{(\omega t - kl_1) + (\omega t - kl_2)}{2} = \frac{2\omega t - k(l_1 + l_2)}{2} = \omega t - k\frac{l_1 + l_2}{2}.$$

   - For the second cosine term:

$$\frac{(\omega t - kl_1) - (\omega t - kl_2)}{2} = \frac{-k(l_1 - l_2)}{2} = k\frac{l_2 - l_1}{2}.$$

4. **Substitute back into the expression:**

$$\cos(\omega t - kl_1) + \cos(\omega t - kl_2) = 2\cos\left(\omega t - k\frac{l_1 + l_2}{2}\right)\cos\left(k\frac{l_2 - l_1}{2}\right).$$

5. **Multiply by A to get R:**

$$R = A\left[\cos(\omega t - kl_1) + \cos(\omega t - kl_2)\right] = 2A\cos\left(\omega t - k\frac{l_1 + l_2}{2}\right)\cos\left(k\frac{l_2 - l_1}{2}\right).$$

Conclusion

Thus, we have shown that:

$$R = A\left[\cos(\omega t - kl_1) + \cos(\omega t - kl_2)\right] = 2A\cos\left(\omega t - k\frac{l_2 + l_1}{2}\right)\cos\left(k\frac{l_2 - l_1}{2}\right).$$